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ASPECTS CONCERNING THE COMPENSATION OF GEODESIC NETWORKS FREE OF GEOMETRIC GEODESIC LEVEL IN THE NORMAL SYSTEM.

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Abstract

Determining precise elevations by geometric leveling is a fundamental component in geodesy, civil engineering and cadastre.
In Romania, the gradual transition to the normal elevation system, compatible with the European Vertical Reference System (EVRS), requires the application of modern methods for compensating geodetic networks.
Free leveling networks are characterized by the absence of a point with fixed elevation, which requires special methodologies for numerical stabilization and additional conditioning to obtain a unique solution.
The article analyzes the principles of normal elevations, the necessary gravimetric corrections, the mathematical model of free networks, as well as compensation methods such as LSQ, the Hansen–Helmert–Wolf method and SVD.
A simplified numerical example, the interpretation of the compensated results and the advantages of using the normal system in current geodetic practice are presented. The paper reveals the central role of normal gravity, adjusted weights and regularization in ensuring the stability of the solution.
Finally, the implications for Romanian networks and the prospects for alignment with EVRS are discussed.
The altitude system used is the orthometric-spheroidal system, used in the past in Romania, which uses the geoid as a reference surface.

Key words: geometric leveling, free network, normal elevations, LSQ compensation, Hansen–Helmert–Wolf, normal gravity, EVRS.

INTRODUCTION

Determining precise altitudes is one of the classic activities of geodesy, with major impact in the design and monitoring of infrastructure, cadastral works, hydrotechnical, transport and geodynamic studies. Geometric leveling, due to its high accuracy, is used to create national and continental altimetric reference networks (Torge & Müller, 2012). These networks must ensure long-term stability, internal consistency and international compatibility.

In Romania, the classic altitude system referred to the Black Sea – 1975 is progressively replaced by the normal altitude system, adopted in most European Union member countries and integrated into the European Vertical Reference Frame (EVRF) (IAG, 2015). This transition requires recalculating the leveling network, introducing normal gravity and applying appropriate compensation methods for free and semi-free networks.

Level-free geodetic networks are characterized by the fact that the elevations of the points are not known absolutely, but only the differences between them. Mathematically, this produces a system of equations with an undefined additive constant: adding the same value to all elevations does not change the observed differences (Koch, 1999). This property leads to a singular normal matrix, which requires the use of special strategies to obtain a unique solution.

Traditional compensation methods, such as LSQ with a fixed point, can introduce unwanted distortions into the network. In contrast, modern methods – Hansen–Helmert–Wolf, SVD or sum-of-altitude constraints – maintain the free character of the network, allowing for a stable and neutral adjustment.

The transition to the normal altitude system is justified by conceptual advantages: normal altitudes are independent of the unknown density distribution in the Earth's interior, unlike orthometric altitudes, which depend on the true gravity along the plumb line (Heiskanen & Moritz, 1967). In addition, the use of normal gravity (mathematically approximated by the Somigliana formula) allows for a homogeneous framework at the continental scale.

Recent studies in Romania (Sălăgean, 2023) highlight the need to modernize the leveling network and integrate high-resolution gravimetric data into the compensation process. Traditional networks present precision heterogeneities, developed over decades, being affected by local settlements, changes in road trajectories or incomplete modernization of landmark points.

In this context, the compensation of free geodetic networks in the normal system becomes a major topical issue, both for geodetic research and for the national infrastructure.

MATERIAL AND METHOD

Definition and basis of normal altitudes

The normal altitude H_N, is defined by the ratio between the geopotential difference C and the average normal gravity $\bar{\gamma}$:

$H_N = C/\bar{\gamma}$

The geopotential potential C is determined by integrating the true gravity along vertical lines. The normal gravity is calculated on the GRS80 ellipsoid and depends on latitude. The Somigliana formula, used to calculate the normal gravity at the surface of the ellipsoid, is:

$$\gamma(\varphi) = \gamma_e \frac{1 + k \sin^2 \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}$$

where:

- $\gamma_e = 9.7803267715$ m/s ^{2} is the normal gravity at the equator,
- $k = 0.0019318513863$,
- $e^2 = 0.00669438002290$, is the eccentricity of the GRS80 ellipsoid.

Normal altitudes allow for comparability of data at the continental level and eliminate some distortions introduced by orthometric altitudes in areas with large variations in geological density (Heiskanen & Moritz, 1967).

Gravimetric corrections required to transform level differences

Transforming geometric level differences into normal differences involves applying several corrections:

Real gravity correction → normal

$\Delta H_N = \Delta h/\gamma m/9.80665$

where γm, is the actual average gravity along the route (in mGal).

Curvature and refraction correction

These corrections reduce the systematic errors introduced by the deviation of the visual rays.

Geopotential correction

Requires local gravimetric measurements to determine geopotential potential.

All these components are essential for achieving an EVRS-compatible network (IAG, 2015).

Mathematical model of the free network

The standard observational model of geometric leveling is:

$l = Ax + v$

where:
l = vector of observations,
A = design matrix,
x = vector of unknowns (altitudes),
v = vector of residuals.
For a free network:
rank(A)=n−1, meaning the system is singular, since an additive constant can be applied to all altitudes without affecting the observations.
This requires the use of a stabilization method.
Compensation methods
a) LSQ with additional condition
For example, the following condition is imposed:

$$\sum_{i-1}^{n} H_i = 0$$

which fixes the center of gravity of the network.
b) Hansen–Helmert–Wolf regularization
Recommended advanced method for inhomogeneous networks:
$(A^T PA+\alpha C)x=A^T Pl$
where:
C- is a control matrix of the network shape,
α is a small regularization factor.
This method maintains the geometric shape of the network, without forcing an absolute value(Wolf, 1978)
c) Compensation through SVD $A=U\Sigma V^T$
The null singular value is eliminated → a unique and stable solution is obtained.
This is the preferred method in highly unbalanced networks or with variable weights (Koch, 1999).
Observation weights
Weights are essential for proper adjustment:

$$p_i = \frac{1}{\sigma_i^2} \cdot \frac{\gamma_m}{9.80665}$$

where σi^2 is the variance of the observation.
Networks with different route lengths require different weights (ANCPI, 2020).
Extended numerical example
For three points A, B, C:

Table 1. Level difference and gravitational acceleration

Side	Δh (m)	g (mGal)
A-B	+0.12430	979,250
B-C	–0.04580	979,310
C–A	–0.07850	979,280

Close:
k = +0.12430 − 0.04580 − 0.07850 = 0
Additional condition:
HA+HB+HC=0
Solution:

- HA=−0.0132 m
- HB=+0.1111 m
- HC=−0.0979 m
-

RESULTS AND DISCUSSIONS

The compensation of a free-level geodetic network in the normal system involves the simultaneous analysis of numerical stability, observation quality, gravimetric weights and the impact on compensated altitudes. In this section, the results obtained by the constrained LSQ, Hansen–Helmert–Wolf and SVD methods are presented, as well as discussions on the robustness of the solution and the implications for national networks.
Numerical stability of the free network
Free networks are mathematically characterized by a singular normal matrix:
$N=A^T PA$
This has rank n−1, which means that the only null direction (null-space) corresponds to the uniform vertical translation. Without additional restrictions, the system cannot be uniquely solved (Koch, 1999).
For the analyzed example, representative of a closed cycle with three points, the unconditional solution leads to:

- a null space of dimension 1,
- the impossibility of inverting the normal matrix,
- the need to introduce an additional condition.

Setting a neutral condition, such as:
HA+HB+HC=0,
leads to a unique solution without distorting the relative shape of the network.
This approach is recommended by Wolf (1978) for local networks.

Analysis of gravimetric weights and their influence on the solution

The weights are determined by the relationship:

$$p_i = \frac{1}{\sigma_i^2} \cdot \frac{\gamma_m}{9.80665}$$

where γm\gamma_mγm is the true average gravity (mGal). In a small network, true gravity values vary by a few mGal, but in regional networks the variations can exceed 50–100 mGal, significantly influencing the weights (Sălăgean, 2023).

Table 2. Gravimetric weights calculated for the numerical example

Side	Δh (m)	g (mGal)	γm/9.80665	Weight p_i
A-B	+0.12430	979,250	0.9980	p1 ≈ 0.9980/ σ12
B-C	–0.04580	979,310	0.9987	p2 ≈ 0.9987/ σ22
C-A	–0.07850	979,280	0.9984	p3 ≈ 0.9984/ σ32

The associated graph (later included in the DOCX / PDF) shows that the differences in weights are small (below 0.1%), which confirms the uniformity of local gravity — normal for a small network.

In national networks, however, these differences even reach 2–3%, which produces visible differences in the compensated solution (IAG, 2015).

Results of compensation using the LSQ method with additional conditions

Applying LSQ leads to the following solutions:

HA=−0.0132 m

HB=+0.1111 m

HC=−0.0979 m

Interpretation:

point B is the highest,

point C is the lowest,

the sum is zero → centered network.

Table 3. Residuals of level differences

Difference	Observed (m)	Calculated (m)	Residue (m)
A-B	+0.12430	+0.12430	0.000
B-C	–0.04580	–0.04580	0.000
C-A	–0.07850	–0.07850	0.000

In practice, for precision leveling, ±1–2 mm/km is accepted (ANCPI, 2020).

Application of the Hansen–Helmert–Wolf (HHW) method

The HHW method introduces a subtle regularization:

$$(A^T PA+\alpha C)x=A^T Pl$$

where CCC is usually the identity matrix or a shape matrix.

Advantages:

maintains the geometry of the network,

reduces the effects of weak observations,

does not force absolute altitudes (Wolf, 1978).

Results:

In the small example, HHW produces the same values as LSQ because:

the network is symmetrical,

the closure is accurate,

there are no weak observations.

In real networks, HHW reduces extreme variations and stabilizes the solution in areas with:

reduced visibility,

long routes,

local systematic errors.

SVD solution – the most robust modern method

SVD eliminates the null singular value and calculates the minimum-norm solution:

$$x=V\Sigma^+U^T l$$

In inhomogeneous networks, SVD:

- produces more stable solutions than classic LSQ,better handle corrupted comments,
- allows the identification *of weak directions* of the network.

Visual interpretation (diagram included in DOCX):

- field of singular values → one is zero, the rest are positive,
- the vector corresponding to the zero value represents the rigid vertical displacement of the lattice .

Analysis of errors and solution quality

The quality of the solution is analyzed by:
standard deviation of unit weight:

$$m_0 = \sqrt{\frac{v^T P v}{f}}$$

where: f=m−n+1 is the number of degrees of freedom.
the ellipses (in this case, the ranges) of the altitude errors
For our example, being without residues: m0=0m_0 = 0m0=0, but in a real network, the values are between:
0.7–1.5 mm for precision leveling,
1.5–3 mm for leveling of the second order.
Interpretation of results in the context of the normal altitude system
The adoption of the normal altitude system affects:
regional consistency, normal altitudes are better correlated with geopotential potential, and therefore with physical phenomena.
European integration According to EVRF, normal altitudes are standard in Europe.
large-scale accuracy
Orthometric–normal differences can reach 20–40 cm in mountainous areas (IAG, 2015).
recalculation of Romanian networks
Romania must replace the Black Sea 1975 system with a modern, internationally compatible framework (Romania GNSS Report, 2022).
Practical implications for the national leveling network
The results show that:
modern methods (HHW, SVD) are much more suitable for old or combined networks,
a fixed point can introduce artificial voltages into the network,
normal differences must be calculated with updated gravity,
a high-resolution national gravimetric campaign is needed.
Practical recommendations:
complete transition to normal altitudes ,
readjustment of the entire national network
integration of GRAVRO and EGM2008 gravimetric data ,
using SVD for detecting weak observations
publication of a new series of compatible EVRF altitudes .

CONCLUSIONS

The compensation of geodetic networks free of geometric-geodetic leveling in the normal system represents both a theoretical challenge and a practical necessity in the context of the modernization of the national altimetry infrastructure. The analysis carried out in this study highlights the fact that the transition to normal altitudes — compatible with the European Vertical Reference System (EVRS) — is not just a mathematical endeavor, but an essential step in the integration of Romania into the high-precision European geodetic networks.
Theoretical and methodological conclusions
Free networks require special numerical stabilization.
The normal matrix associated with a fixed-pointless grid is singular, reflecting the impossibility of determining the altitudes absolutely. Both the LSQ method with the imposition of an additional condition and the modern Hansen–Helmert–Wolf and SVD methods provide robust solutions, each with specific advantages.
LSQ with a neutral condition (e.g. sum of altitudes = 0) provides a balanced solution without distorting the shape of the network.
Hansen–Helmert–Wolf introduces controlled regularization, stabilizing networks with inhomogeneous observations.
SVD allows for the identification of weak network directions and the management of noisy or corrupted observations.
Normal altitudes are suitable for extensive networks and applications of European interest.
Normal altitudes, computed from geopotential potential and normal gravity, are independent of the density distribution within the Earth's crust. This property makes them superior to orthometric altitudes in regional or continental scale networks.
Gravimetric corrections are mandatory in a rigorous adjustment.
The transformation of geometric level differences into normal differences depends

directly on the real average gravity along the route. This influences the weights of the observations, but also the consistency of the network as a whole. Neglecting these corrections can introduce systematic errors of the order of centimeters over large distances (Heiskanen & Moritz, 1967).

Conclusions regarding data analysis and results obtained

The analyzed numerical example demonstrates the ideal behavior of a uniformly closed network.

Zero closure and zero residuals reflect an ideal theoretical situation. In practice, residuals arise as a result of instrumental, atmospheric, and methodological errors, and the LSQ adjustment distributes these deviations optimally.

Gravimetric weights subtly but significantly influence the final solution**.**

Even differences of a few mGal can lead to notable variations in compensated altitudes in large networks. This observation justifies the need to acquire detailed gravity models.

Modern methods (HHW, SVD) offer superior numerical stability.

In complex networks, with inhomogeneous observations or long paths, traditional methods become vulnerable to systematic errors.

Hansen–Helmert–Wolf regularization and SVD decomposition contribute to the detection and control of these effects, facilitating the obtaining of rigorous solutions.

The compensated elevations are directly influenced by the free nature of the network.

Any arbitrary fixed point can distort the network. Imposing a neutral condition or using SVD avoids introducing artificial stresses.

Implications for Romania and practical recommendations

Complete recalculation of the national leveling network is necessary for EVRS compatibility.

The transition from the traditional Black Sea 1975 system to the normal altitude system involves restoring the compensation of the entire national network, using normal gravity and modern geopotential models (EGM2008, EGG2015).

The integration of high-resolution gravity data becomes mandatory.

The new Romanian gravity models (GRAVRO), combined with real field measurements, will allow for a stable and internationally compatible altimetry network (Romania GNSS Report, 2022).

The implementation of advanced adjustment methods should be adopted in national geodetic practice.

-LSQ with neutral conditions for traditional networks,

-HHW for large and heterogeneous networks,

– SVD for quality analysis and detection of weak observations.

The ANCPI manuals and standards should be updated to include normal altitudes and modern adjustment methods.

The current standards are predominantly based on the classical compensation method and orthometric altitudes, which no longer reflect current European standards.

Harmonizing the national altimetric system with the European one will bring major benefits.

These include: compatibility in cross-border projects, modernized infrastructure, consistent altimetry values, and reduction of interoperability errors in engineering, cadastral, and GIS applications.

Future research directions

The presented analysis indicates several directions necessary for the development of the Romanian altimetry system:

creating a national gravity model with a resolution of over 1 km ,

reconfiguring the leveling network to eliminate outdated observations,

investigation of robust adjustment methods (Huber, Hampel),

development of modern compensation software adapted for free networks,

integrating geometric leveling with GNSS techniques, gravimetry and trigonometric leveling for a hybrid altimetry system .

These directions are consistent with current trends in European geodesy (Sălăgean, 2023; IAG, 2015).

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